

Exact Solutions of Poiseuille Flow in Porous Media

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Abstract

We find analytical solutions of the Navier-Stokes equation for the flow of an incompressible, Newtonian fluid through a porous medium in channels with a rectangular and a circular cross section. Nield and others have developed an analytical solution for the first case. We express it in an equivalent but a more conventional form. In the second case, we find an exact solution when then Forchheimer term can be ignored and approximate solution when it cannot be ignored.

1 Introduction

The flow of an incompressible, Newtonian fluid through a porous medium of porosity ϵ and permeability K is described by the equation [Zaoli2002]

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} \left(\frac{u_i}{\epsilon} \right) = - \frac{\partial}{\partial x_j} \left(\frac{p\epsilon}{\rho} \right) + \nu_e \Delta u_i + F_i, \quad (1)$$

where ρ is the density of the fluid, u_i is the i th component of its velocity, p the pressure, ν_e the effective viscosity, ν the kinematic viscosity, Δ denotes the Laplace operator and the total force is

$$F_i = - \frac{\epsilon \nu}{K} u_i - \frac{\epsilon F_0}{\sqrt{K}} (u_j u_j)^{1/2} u_i + F_{bi}. \quad (2)$$

It is a sum of the body force F_{bi} and the effect of the porosity of the medium. The geometric function F_0 in equation (2) depends only on the porosity and the nature of the solid particle matrix of the porous medium. Einstein summation convention is used in equation (2) and others in this paper. For a fully developed and steady flow, the left hand side of equation (1) vanishes. Further, if the flow

is driven only by a constant pressure gradient and not an external body force, then equation (1) becomes

$$\nu_e \Delta u_i - \frac{\epsilon \nu}{K} u_i - \frac{\epsilon F_0}{\sqrt{K}} (u_j u_j)^{1/2} u_i + \epsilon G_i = 0, \quad (3)$$

where $\rho G_i = -\partial p / \partial x_i$. In this paper we solve equation (1) in two cases - (a) flow between parallel solid plates with a porous medium between them and (b) Poiseuille flow in a cylindrical channel stuffed with a porous medium. Nield and others [Nield1996] have given an analytical solution of the first case. It expresses the coordinate y as an elliptic integral in the velocity u . We shall develop an equivalent solution, in a more conventional form, in which u is expressed as a Weierstrassian elliptic function of y . We give an exact solution in the second case when the term quadratic in u , called the Forchheimer term [Zaoli2002], is ignored. If the Forchheimer term is included, we develop an approximate analytical solution at the centre of the channel and near its walls.

2 Flow between two infinite parallel plates

Consider a steady, fully-developed flow of a fluid along the x_1 axis between two infinite parallel plates a distance H apart. Let the planes be defined by the equations $x_2 = 0$ and $x_2 = H$. If the channel is long enough that the end effects are negligible along most of the channel's length then we can express the fluid's velocity along the x_1 axis as $u(x_2)$. Equation (3), in this case, becomes

$$\frac{\nu_e}{\epsilon} u'' - \frac{\nu}{K} u - \frac{F_0}{\sqrt{K}} u^2 + \epsilon G = 0, \quad (4)$$

where u'' denotes the second derivative with respect to x_2 and $\rho G = -\partial p / \partial x_1$. Since the channel is bounded by solid walls, the usual no-slip boundary conditions, $u(x_2 = 0) = 0$ and $u(x_2 = H) = 0$, apply. Let U be the magnitude of velocity at the centre of the channel. We convert equation (4) in a non-dimensional form using the relations,

$$u_\star = \frac{u}{U} \quad (5)$$

$$x_{2\star} = \frac{x_2}{H} \quad (6)$$

$$G_\star = \frac{HG}{U^2} \quad (7)$$

and the dimensionless quantities

$$Re = \frac{HU}{\nu} \quad (8)$$

$$Da = \frac{K}{H^2} \quad (9)$$

$$J = \frac{\nu_e}{\nu}. \quad (10)$$

Re , Da and J are the Reynolds number, the Darcy number and the viscosity ratio of the flow. Dropping the \star subscript for sake of notation clarity, we get,

$$u'' + \frac{\epsilon Re G}{J} - \frac{\epsilon}{J Da} u - \frac{F_0 \epsilon Re}{J \sqrt{Da}} u^2 = 0 \quad (11)$$

Introduce a new variable $u_1(x_2)$,

$$u_1(x_2) = \sqrt{\frac{F_0 \epsilon Re}{J \sqrt{Da}}} u(x_2) + \frac{1}{2} \sqrt{\frac{\epsilon}{J F_0 Re (Da)^{3/2}}}, \quad (12)$$

so that equation (11) becomes,

$$u_1'' + 2b^2 - \frac{a^3}{6} u_1^2 = 0, \quad (13)$$

where

$$\frac{a^3}{6} = \sqrt{\frac{F_0 \epsilon Re}{J \sqrt{Da}}} \quad (14)$$

$$2b^2 = \frac{a^3}{6} \left(\frac{\epsilon Re G}{J} + \frac{\epsilon}{4 J F_0 Re (Da)^{3/2}} \right). \quad (15)$$

Multiplying equation (13) by u_1' , the first derivative of $u_1(y)$ with respect to y , and integrating we get,

$$(u_1')^2 + 4b^2 u_1 - \frac{a^3}{9} u_1^3 + k_0 = 0, \quad (16)$$

where k_0 is a constant of integration. Introduce yet another variable $u_2(y) = (a^3/36)u_1(y)$ to transform equation (16) to

$$(u_2')^2 = 4u_2^3 - \frac{b^2 a^3}{9} u_2 - \frac{k_0 a^6}{36^2}. \quad (17)$$

This equation is of the form,

$$(u_2')^2 = 4u_2^3 - g_2 u_2 - g_3, \quad (18)$$

where

$$g_2 = \left(\frac{b^2 a^3}{9} \right) \quad (19)$$

$$g_3 = \frac{k_0 a^6}{36^2}. \quad (20)$$

Its solution is $u_2(x_2) = \wp(x_2 + k_1; g_2, g_3)$ [Whittaker1927], where \wp is the Weierstrassian elliptic function and k_1 is another constant of integration. Therefore, the solution of equation (4) is,

$$u(x_2) = \frac{6J\sqrt{Da}}{F_0 \epsilon Re} \wp \left(x_2 + k_1; \left(\frac{b^2 a^3}{9} \right), \frac{k_0 a^6}{36^2} \right) - \frac{1}{2F_0 Re \sqrt{Da}}. \quad (21)$$

We find the constants k_0 and k_1 using the boundary conditions.

2.1 Finding k_0

We shall follow the method suggested by Nield and others [Nield1996] to determine k_0 . It depends on the fact that at the centre of the channel, the velocity of the fluid is an extremum (actually, a maximum) and hence its first derivative with respect to x_2 vanishes. From the relation between $u_2(x_2)$ and $u(x_2)$, it follows that whenever $u'(x_2) = 0$, $u'_2(x_2)$ also vanishes. In that case, from equation (18), it follows that $4u_2^3 - g_2u_2 - g_3 = 0$. If e_1 , e_2 and e_3 are the roots of this cubic polynomial then at the centre of the channel, $(u_2(1/2) - e_1)(u_2(1/2) - e_2)(u_2(1/2) - e_3) = 0$. Therefore, one of e_1 , e_2 or e_3 is definitely equal to $u_2(1/2)$, the speed at the centre of the channel. Let us, without loss of generality, assume that $e_2 = u_2(1/2)$.

Now, equation (18) can be expressed as,

$$u'_2 = \sqrt{4u_2^3 - g_2u_2 - g_3}, \quad (22)$$

or

$$\int_0^{1/2} dx_2 = \int_{P_0}^{e_2} \frac{du_2}{\sqrt{4u_2^3 - g_2u_2 - g_3}}, \quad (23)$$

where $P_0 = u_2(0)$. Using the relation between u_2 and u and the boundary condition $u(0) = 0$, it is easy to evaluate P_0 to be $\epsilon/(12JDa)$. This equation can still not be solved because both e_2 and g_3 are unknowns. They are however related by the following properties of roots of cubic equations,

$$e_1 + e_2 + e_3 = 0 \quad (24)$$

$$e_1e_2 + e_2e_3 + e_3e_1 = \frac{-g_2}{4} \quad (25)$$

$$e_1e_2e_3 = \frac{g_3}{4} \quad (26)$$

From these equations, we can easily conclude that $g_3 = (4e_2^2 - g_2)e_2$. We can now solve equation (23) numerically and get e_2 , and hence g_3 and k_0 , for given values of ϵ , Re , Da , G and J .

2.2 Finding k_1

We used the boundary condition $u(0) = 0$ and the fact that in a Poiseuille flow, u has a maximum at the centre of the channel to find k_0 . We will now use the second boundary condition $u(1) = 0$ to find k_1 . The equation $u(1) = 0$ is equivalent to,

$$\wp(1 + k_1; g_2, g_3) = P_0 = \frac{\epsilon}{12JDa} \quad (27)$$

This equation always has a number of roots equal to the order of the elliptic function [Whittaker1927], 2 in this case. Since u_2 obeys the differential equation,

$$(\wp'(z; g_2, g_3))^2 = 4\wp^3 - g_2\wp - g_3, \quad (28)$$

where z denotes a complex variable, we can write,

$$dz = \frac{d\wp}{\sqrt{4\wp^3 - g_2\wp - g_3}}. \quad (29)$$

Further, since the Weierstrassian elliptic function has a pole of order 2 at $z = 0$, we can integrate the left hand side of this equation from $\wp = \infty$ to $\wp = P_0$. The corresponding limits of integration on the right hand side are 0 and P_0 , respectively. Therefore,

$$z(\wp = P_0) - z(\wp = \infty) = z(\wp = P_0) = \int_{\wp=\infty}^{\wp=P_0} \frac{d\wp}{\sqrt{4\wp^3 - g_2\wp - g_3}}, \quad (30)$$

or

$$1 + k_1 = \int_{\wp=\infty}^{\wp=0} \frac{d\wp}{\sqrt{4\wp^3 - g_2\wp - g_3}} - \int_{\wp=P_0}^{\wp=0} \frac{d\wp}{\sqrt{4\wp^3 - g_2\wp - g_3}}, \quad (31)$$

that is,

$$k_1 = -1 - \int_{\wp=0}^{\wp=\infty} \frac{d\wp}{\sqrt{4\wp^3 - g_2\wp - g_3}} + \int_{\wp=0}^{\wp=P_0} \frac{d\wp}{\sqrt{4\wp^3 - g_2\wp - g_3}} \quad (32)$$

The second term on the right hand side is the value of zero of Weierstrassian elliptic function. We refer to the papers by Eichler and Zagier [Eichler1982] or Duke and Imamoglu [Duke2008] for the formulae to evaluate it. The third term on the right hand side is an elliptic integral that can be evaluated numerically.

3 Poiseuille flow without Forchheimer term

Consider the flow of a viscous, Newtonian, incompressible fluid through a cylindrical pipe of radius R , stuffed with a porous medium of porosity ϵ and permeability K . Let us, for sake of simplicity, ignore the Forchheimer term which is quadratic in velocity. We can do so, if we assume a slow flow. Continuing the notation of the previous section, the equation of a fully developed flow is now given by (see equation (3))

$$\nu_e \Delta u_i - \frac{\epsilon \nu}{K} u_i + \epsilon G = 0, \quad (33)$$

Although this is a linear partial differential equation, the author is not aware of prior work giving its solution for the problem at hand.

Let us use a cylindrical coordinate system (r, ϕ, z) with the z axis coinciding with the axis of the cylindrical channel. A fully developed flow along the cylinder is described by the velocity function $u(r)\mathbf{e}_z$, \mathbf{e}_z be a unit vector along the z axis. Equation (33) then becomes,

$$\nu_e \left(\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} \right) - \frac{\epsilon \nu}{K} u + \epsilon G = 0, \quad (34)$$

If U is the velocity at the centre of the channel, we can introduce non-dimensional variables using the relations,

$$u_{\star} = \frac{u}{U} \quad (35)$$

$$r_{\star} = \frac{r}{R} \quad (36)$$

$$G_{\star} = \frac{RG}{U^2} \quad (37)$$

and the dimensionless quantities

$$Re = \frac{RU}{\nu} \quad (38)$$

$$Da = \frac{K}{R^2} \quad (39)$$

$$J = \frac{\nu_e}{\nu}. \quad (40)$$

Re , Da and J are the Reynolds number, the Darcy number and the viscosity ratio of the flow. Dropping the \star subscript for sake of notation clarity, we get,

$$u'' + \frac{u'}{r} + c_0 - a_0^2 u = 0, \quad (41)$$

where primes now denote differentiation with respect to r and the constants a_0 and c_0 are,

$$a_0 = \sqrt{\frac{\epsilon}{JDa}} \quad (42)$$

$$c_0 = \frac{\epsilon ReG}{J} \quad (43)$$

We will solve equation (41) with the boundary conditions $u(r = R) = 0$ and that $u(r)$ is finite for all $0 \leq r \leq R$. Let $u_1(r) = c_0 - a_0^2 u(r)$. Equation (41) transforms to,

$$u_1'' + \frac{u_1'}{r} - a_0^2 u_1 = 0, \quad (44)$$

while the boundary condition becomes $u_1(r = R) = c_0$. Equation (44) is the modified Bessel equation of order 0 whose solution is $k_0 I_0(a_0 r) + k_1 K_0(a_0 r)$. Here, k_0 and k_1 are constants of integration, while I_0 and K_0 are modified Bessel functions of order 0. Since u is finite at $r = 0$, $k_1 = 0$, while $u_1(r = R) = c_0$ implies $k_0 = c_0 / I_0(a_0 R)$. Therefore, the solution of (44) is,

$$u_1(r) = \frac{c_0}{I_0(a_0 R)} I_0(a_0 r) \quad (45)$$

The solution of (41), therefore, is,

$$u(r) = ReGDa \left(1 - \frac{I_0(a_0 r)}{I_0(a_0 R)} \right), \quad (46)$$

where we have used a_0 defined in equation (42) to simplify notation.

3.1 Asymptotic form of solution

We shall now examine the behaviour of the solution (46) as the porosity of the medium tends to 1, that is in the limiting case of no porous medium. To that end, write (46) as,

$$u(r) = \frac{ReGDa}{I_0(a_0R)} (I_0(a_0R) - I_0(a_0r)), \quad (47)$$

The series form of the modified Bessel function of order m is (formula 9.6.10 of [Abramowitz1964]),

$$I_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{k=\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(m+k+1)} \quad (48)$$

Therefore,

$$u(r) = \frac{ReGDa}{I_0(a_0R)} \left(\left(\frac{\epsilon}{JDa}\right) \frac{(R^2 - r^2)}{4\Gamma(2)} + \left(\frac{\epsilon}{JDa}\right)^2 \frac{(R^4 - r^4)}{4^2 2! \Gamma(3)} + \dots \right), \quad (49)$$

The absence of porous medium is characterised by the limits $\epsilon \rightarrow 1$, $J \rightarrow 1$ and $Da \rightarrow \infty$, or in terms of our constants, $a_0 \rightarrow 0$.

$$\lim_{a_0 \rightarrow 0} u(r) = \frac{ReG}{4} (R^2 - r^2), \quad (50)$$

which is the solution of equation (41) in the limits $a_0 \rightarrow 0$ and $\epsilon \rightarrow 1$.

4 Poiseuille flow with Forchheimer term

We shall now consider the Forchheimer term in the equation of flow. That is, we will attempt to solve the equation,

$$\nu_e \Delta u - \frac{\epsilon \nu}{K} u - \frac{F_0}{\sqrt{K}} u^2 + \epsilon G = 0, \quad (51)$$

subject to the boundary condition $u(r = R) = 0$. After introduction of non-dimensional variables as before, we can transform it to,

$$u'' + \frac{u'}{r} + c_0 - a_0^2 u - b_0 u^2 = 0, \quad (52)$$

where the constants a_0 and c_0 are same as before while b_0 is defined as,

$$b_0 = \frac{F_0 Re}{J \sqrt{Da}}. \quad (53)$$

Equation (51) is similar to equation (41) except for the term $b_0 u^2$. It is also similar to equation (11) except for the additional term u'/r . However, unlike

equations (41) and (11), it is not easy to obtain a solution of this equation. We shall therefore attempt an approximation. We shall use the solution of the problem without the Forchheimer term as a guide to approximation. In particular, we shall examine the behaviour of u'/r and u'' at the centre of the channel and at its walls. From equation (46),

$$u(r) = (ReGDa) \left(1 - \frac{I_0(a_0 r)}{I_0(a_0 R)} \right) \quad (54)$$

$$u'(r) = -\frac{(ReGDa)a_0}{I_0(a_0 R)} I_1(a_0 r) \quad (55)$$

$$u''(r) = -\frac{(ReGDa)a_0^2}{I_0(a_0 R)} \left(I_0(a_0 r) - \frac{I_1(a_0 r)}{a_0 r} \right), \quad (56)$$

These relations follow from,

$$\frac{d}{dx}(x^p I_p(x)) = x^p I_{p-1}(x) \quad (57)$$

$$I_{-p}(x) = I_p(x) \quad (58)$$

From the series representation of $I_m(x)$, we can easily show that,

$$\lim_{r \rightarrow 0} \frac{u''(r)}{(u'(r)/r)} = 1 \quad (59)$$

$$\lim_{r \rightarrow R} \frac{u''(r)}{(u'(r)/r)} = \left(a_0 R \frac{I_0(a_0 R)}{I_1(a_0 R)} - 1 \right) \quad (60)$$

Typically, a_0 is a large number (because Da is usually very small). Therefore equation (60) suggests that near the walls of the channel, the second derivative of velocity dominates $u'(r)/r$. However, at the centre of the channel, both terms are of equal magnitude.

4.1 Approximate solution near the channel's wall

We can ignore the u'/r term near the wall of the channel. Equation (52) can then be approximated as

$$u'' + c_0 - a_0^2 u - b_0 u^2 = 0 \quad (61)$$

A transformation $u_1 = u\sqrt{b_0} + a_0^2/(2\sqrt{b_0})$ gives,

$$u_1'' + \left(c_0 + \frac{a_0^4}{4b_0} \right) u_1 - u_1^2 = 0 \quad (62)$$

Multiplying by u_1' , and integrating with respect to r gives,

$$(u_1')^2 + 2 \left(c_0 + \frac{a_0^4}{4b_0} \right) u_1 - \frac{2}{3} u_1^3 + k_0 = 0, \quad (63)$$

where k_0 is a constant of integration. Introducing $u_2 = 6u_1$ gives,

$$(u'_2)^2 = 4u_2^3 - 12 \left(c_0 + \frac{a_0^4}{4b_0} \right) u_2 - 36k_0, \quad (64)$$

whose solution is a Weierstrassian elliptic function $\wp(r + k_1; g_2, g_3)$, with

$$g_2 = 12 \left(c_0 + \frac{a_0^4}{4b_0} \right) \quad (65)$$

$$g_3 = 36k_0 \quad (66)$$

and k_1 being a constant of integration. The approximate solution at the centre of the channel is,

$$u(r) = \frac{12}{\sqrt{b_0}} \wp \left(r + k_1; 12 \left(c_0 + \frac{a_0^4}{4b_0} \right), 36k_0 \right) - \frac{a_0^2}{2b_0} \quad (67)$$

The constant k_1 can be computed in a manner similar to section 2. However, k_0 will have to be chosen so to adjust the solution near the wall to that mid-way between centre and the wall.

4.2 Approximate solution at the centre of the channel

We observed that near the centre of the channel, the second derivative of the velocity is of the same order of magnitude as the $u'(r)/r$ term. Therefore, we assume that the equation of flow is approximated by

$$2 \frac{u'}{r} + c_0 - a_0^2 u - b_0 u^2 = 0 \quad (68)$$

Using the transformation $u_1 = u\sqrt{b_0} + a_0^2/(2\sqrt{b_0})$, we get

$$\frac{2u'_1}{r\sqrt{b_0}} + \left(c_0 + \frac{a_0^4}{4b_0} \right) - u_1^2 = 0 \quad (69)$$

It can be further simplified to,

$$\frac{du_1}{u_1^2 - d_0^2} = \frac{\sqrt{b_0}}{2} r dr, \quad (70)$$

where the new constant d_0 is defined as,

$$d_0^2 = \left(c_0 + \frac{a_0^4}{4b_0} \right) \quad (71)$$

Equation (70) can be readily integrated to give,

$$u(r) = -\frac{d_0}{\sqrt{b_0}} \tanh \left\{ \left(\frac{d_0 \sqrt{b_0}}{4} r^2 + k_0 \right) \right\} - \frac{a_0^2}{2b_0}, \quad (72)$$

where k_0 is a constant of integration. There is no boundary condition available at the centre to determine k_0 and like the identically named constant in the previous section, it too will have to be determined by adjusting the solution at the centre to that valid mid-way between centre and the wall.

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